MOTION OF A THIN FLUID LAYER UNDER THE ACTION OF GRAVITY AND SURFACE TENSION FORCES

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Equations describing the motion of a thin layer of an ideal incompressible fluid under the action of gravity and surface tension forces are derived. These equations are a generalization of the well-known equations of the "shallow water" theory [1]. One of their applications is in the study of the motion of a fluid under conditions of weightlessness. Some static and dynamic problems are considered within the framework of the equations derived.

1. We consider the potential flow of an ideal incompressible fluid of density ρ. Let t be time; x, y, z the Cartesian coordinates in the flow region; u, v, w the projections of the fluid velocity on the x, y, z axes; p the pressure; g the constant acceleration due to gravity or other mass forces directed opposite to the z-axis. The equations of motion and continuity and the conditions of irrotational motion may be written as

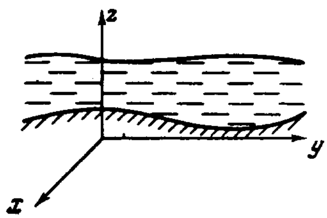

Fig. 1

$$u_t + uu_x + vu_y + wu_z + p_x / \rho = 0,\qquad (1.1)$$

$$v_t + uv_x + vv_y + wv_z + p_y / \rho = 0$$

$$w_t + uw_x + vw_y + ww_z + p_z / \rho + g = 0,$$

$$u_x + v_y + w_z = 0$$

$$w_y = v_z,\ u_z = w_x,\ v_x = u_y$$

(subscripts denote partial derivatives in all cases).

Let the fluid (Fig.1) be bounded below by the stationary floor $z = h(x,y)$ and above by the free surface $z = f(x,y,t)$. Let us write the conditions of impermeability of the floor and the dynamic and kinematic conditions on the free surface,

$$w = h_x u + h_y v \quad \text{for} \quad z = h(x, y) \qquad (1.2)$$

$$p = p_0 - \sigma (1 + f_x^2 + f_y^2)^{-3/2} [f_{xx}(1 + f_y^2) - 2 f_x f_y f_{xy} + f_{yy}(1 + f_x^2)]$$

$$w = f_t + f_x u + f_y v \quad \text{for} \quad z = f(x, y, t)$$

where p_0 is the constant pressure outside the fluid, σ is the coefficient of surface tension, and the factor appearing with σ is the curvature of the free surface; it is assumed that the fluid everywhere lies below the free surface.

We shall follow the pattern of reasoning used in [1] to derive the "shallow water" equations in the absence of surface tension (another derivation is given in [2]). First we substitute variables, functions, and constants,

$$z = \delta\zeta, \quad w = W / \delta, \quad h = \delta H, \quad f = \delta F, \quad g = g' / \delta, \sigma = \sigma' / \delta \qquad (1.3)$$

The dimensionless parameter δ may be defined as the ratio of a characteristic dimension along the z-axis (e.g. the average depth of the fluid) to the characteristic dimension in the xy-plane. Henceforth, it is assumed to be small. Converting to variables (1.3), we reduce Equations (1.1), (1.2) to the form

$$\begin{gathered}
Wu_\zeta + \varepsilon(u_t + uu_x + vu_y + p_x / \rho) = 0 \qquad (\varepsilon = \delta^2) \qquad (1.4)\\
Wv_\zeta + \varepsilon(v_t + uv_x + vv_y + p_y / \rho) = 0\\
WW_\zeta + \varepsilon(W_t + uW_x + vW_y + p_\zeta / \rho + g') = 0\\
W_\zeta + \varepsilon(u_x + v_y) = 0, \quad W_y = v_\zeta, \quad u_\zeta = W_x, \quad v_x = u_y\\
W = \varepsilon(H_x u + H_y v) \quad \text{for} \quad \zeta = H(x, y)\\
p = p_0 - \sigma'(1 + \varepsilon F_x^2 + \varepsilon F_y^2)^{-3/2}[F_{xx}(1 + \varepsilon F_y^2) - 2\varepsilon F_x F_y F_{xy} + F_{yy}(1 + \varepsilon F_x^2)]\\
W = \varepsilon(F_t + F_x u + F_y v) \quad \text{for} \quad \zeta = F(x, y, t)
\end{gathered}$$

The functions sought will be represented as series in the small parameter ε

$$\begin{gathered}
u = u^\circ + \varepsilon u^1 + \ldots, \quad v = v^\circ + \varepsilon v^1 + \ldots, \quad W = W^\circ + \varepsilon W^1 + \ldots\\
p = p^\circ + \varepsilon p^1 + \ldots, \quad F = F^\circ + \varepsilon F^1 + \ldots \quad (\varepsilon \ll 1) \qquad (1.5)
\end{gathered}$$

The coefficients of expansions (1.5) and their derivatives with respect to t, x, y, ζ are considered finite as $\varepsilon \to 0$. We substitute (1.5) into (1.4) and equate coefficients of like powers of ε. From the zeroth approximation we obtain

$$\begin{gathered}
W^\circ \equiv 0, \quad u_\zeta^\circ = v_\zeta^\circ \equiv 0, \quad u_x^\circ = v_y^\circ\\
p^\circ = p_0 - \sigma'(F_{xx}^\circ + F_{yy}^\circ) \quad \text{for} \quad \zeta = F^\circ
\end{gathered} \qquad (1.6)$$

Taking into account (1.5), (1.6) and equating the coefficients of ε in (1.4) we arrive at Equations

$$\begin{gathered}
u_t^\circ + u^\circ u_x^\circ + v^\circ u_y^\circ + p_x^\circ / \rho = 0\\
v_t^\circ + u^\circ v_x^\circ + v^\circ v_y^\circ + p_y^\circ / \rho = 0, \quad p_\zeta^\circ + \rho g' = 0\\
W_\zeta^1 + u_x^\circ + v_y^\circ = 0 \qquad (1.7)\\
W^1 = H_x u^\circ + H_y v^\circ \quad \text{for} \quad \zeta = H, \quad W^1 = F_t^\circ + F_x^\circ u^\circ + F_y^\circ v^\circ \quad \text{for} \quad \zeta = F^\circ
\end{gathered}$$

Henceforth we shall limit ourselves to the first nonzero terms of expansions (1.5). For this reason, in writing (1.7), we have omitted the irrotation conditions and the boundary condition for p which contain u^1, v^1, p^1, F^1. From the equation of continuity and the condition at the floor in (1.7) we have

$$W^1 = (Hu^\circ)_x + (Hv^\circ)_y - \zeta(u_x^\circ + v_y^\circ) \qquad (1.8)$$

Substituting (1.8) into the last condition of (1.7), we obtain an equation for F°

$$F_t^\circ + [(F^\circ - H)\, u^\circ]_x + [(F^\circ - H)\, v^\circ]_y = 0 \tag{1.9}$$

The pressure can be determined from the third equation of (1.7) and the condition on the free surface in (1.6),

$$p^\circ = p_0 - \sigma' \,(F_{xx}{}^\circ + F_{yy}{}^\circ) - \rho g' \,(\zeta - F^\circ) \tag{1.10}$$

Limiting ourselves to the first nonzero terms in (1.5), we set

$$u \approx u^\circ, \quad v \approx v^\circ, \quad w = W \,/\, \delta \approx \delta W^1, \quad p \approx p^\circ, \quad f = \delta F \approx \delta F^\circ$$

Then, taking into account (1.3),(1.10), we make (1.6),(1.7),(1.9) yield the following system of equations for the functions $u\,(x, y, t)$, $v\,(x, y, t)$, $f\,(x, y, t)$:

$$\begin{aligned} &u_t + uu_x + vu_y + gf_x - (\sigma \,/\, \rho)\, \Delta f_x = 0 \\ &v_t + uv_x + vv_y + gf_y - (\sigma \,/\, \rho)\, \Delta f_y = 0 \\ &f_t + [(f - h)\, u]_x + [(f - h)\, v]_y = 0, \qquad u_x = v_y \end{aligned} \tag{1.11}$$

The Laplacean Δ, as well as the operators ∇ , div , rot are considered operative in the xy-plane in all cases. By introducing the two-dimensional velocity vector $\mathbf{v}$ with the components u, v, we can rewrite system (1.11) in the form

$$\begin{aligned} &\mathbf{v}_t + (\mathbf{v}\nabla)\, \mathbf{v} + \nabla\, (gf - \sigma\rho^{-1}\Delta f) = 0 \\ &f_t + \operatorname{div}\, [(f - h)\, \mathbf{v}] = 0, \qquad \operatorname{rot} \mathbf{v} = 0 \end{aligned} \tag{1.12}$$

The functions w , p are expressed in terms of $\mathbf{v}$, f by expressions which follow from (1.8),(1.10),

$$w = \operatorname{div}\, (h\mathbf{v}) - z \operatorname{div} \mathbf{v}, \qquad p = p_0 - \sigma\Delta f - \rho g\, (z - f) \tag{1.13}$$

Equations (1.12) describe the irrotational motion of a fluid under the action of gravity and surface tension forces when the characteristic depth of the fluid is small in comparison with a characteristic dimension (e.g. with the wavelength) in the xy-plane. The angles between the floor $z = h(x, y)$ and the free surface, respectively, and the horizontal plane are likewise assumed to be small, although the amplitude of the waves is arbitrary (the depth can change a finite number of times). System (1.12) is nonlinear, although it is considerably simpler than the initial system (1.1) , since it contains fewer of the functions and independent variables sought.

Equations of long waves with allowance for surface tension are derived in [3] and investigated, for example, in [4]. The equations in [3] are valid only for small-amplitude plane waves propagating in a single direction, and do not degenerate to the "shallow water" equations for $\sigma = 0$. In addition, in deriving his equations, the author of [3] assumes the surface tension forces to be small in comparison with gravity, which obviates the possibility of considering the case of weightlessness $g = 0$.

Equations (1.12),(1.13) were obtained without these limitations and constitute direct generalizations of the equations of the first approximation of the "shallow water" theory [1], into which they degenerate for $\sigma = 0$. For $g = 0$, equations (1.12),(1.13) describe the motion of a thin layer of a weightless fluid.

To find the equations of motion of the fluid it is necessary to solve system (1.12) under certain initial and boundary conditions, and then to determine w and p from Formulas (1.13). Let $D(t)$, generally speaking, be the variable projection of the flow region on the xy-plane, and let the contour Γ be the boundary of the region $D(t)$. As the initial conditions for system (1.12) it is necessary to specify at the initial instant t_0 the function f and the potential solenoidal vector $\mathbf{v}$ in the region $D(t_0)$. If the contour Γ is given, two boundary conditions (one of the velocity and one for the shape of the free surface) must be specified thereon. If it is not given, three conditions are required. For example, let the flow region be bounded by vertical walls (a cylindrical surface with the directrix Γ), and let γ be the contact angle at the boundaru between the fluid and the walls; for applicability of the derived equations, this angle is assumed to be close to $\frac{1}{2}\pi$. The conditions on the contour Γ then become

$$\mathbf{vn} = 0, \qquad \mathbf{n}\nabla f = -\cos\gamma \approx \gamma - {}^1/_2\pi$$

where $\mathbf{n}$ is the inner normal to the contour Γ.

2. As an example, let us consider the problem of equilibrium of a drop on a plane horizontal wall (Fig.2). The contact angle γ between the fluid and wall is considered small $(0 \leqslant \gamma \ll 1)$, and the shape of the drop sufficiently shallow. It is then possible to apply Equations (1.12), which for the case of equilibrium give us

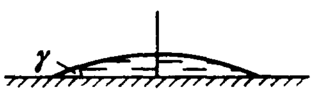

Fig. 2

$$\nabla(\rho g f - \sigma \Delta f) = 0 \tag{2.1}$$

Let us make the xy-plane coincident with the plane of the wall. Let the drop be axisymmetrical and let its base be the circle $x^2 + y^2 = R^2$.

Equation (2.1) then yields

$$[\rho g f - (\sigma / r)(r f')']' = 0$$
$$r = (x^2 + y^2)^{1/2}$$
$$(\ldots)' = d(\ldots)/dr \tag{2.2}$$

for the shape $f(r)$ of the drop surface.

The solution of Equation (2.2) depends on three arbitrary constants which can be determined from conditions

$$f(R) = 0, \quad f'(R) = -\gamma$$

and the condition of boundedness $f(0)$.

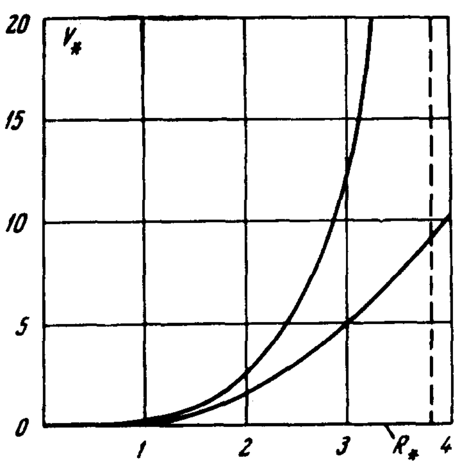

Fig. 3

For $\rho > 0$ the solution is expressed in terms of Bessel functions of an imaginary argument,

$$f(r) = \frac{\gamma\sigma^{1/2}}{(\rho g)^{1/2} I_1(R_*)}\left[I_0(R_*) - I_0\left(r\sqrt{\rho g/\sigma}\right)\right], \qquad R_* = R(\rho g/\sigma)^{1/2} \tag{2.3}$$

With the aid of solution (2.3) it is easy to compute the volume v of the drop. The relationship betwenn the dimensionless volume v_* and dimensionless radius R_* of the drop may be written as

$$v_* = \frac{R_*^2 I_0(R_*) - 2R_* I_1(R_*)}{I_1(R_*)} \qquad \left(v_* = \frac{v\,(\rho g)^{3/2}}{\pi\gamma\sigma^{3/2}}\right) \tag{2.4}$$

This dependence is depicted by the lower curve in Fig.3 and can serve for determining the radius of the drop from a given volume. The following asymptotic formulas hold for both small and large R_* :

$$v_* \sim {}^1/_4 R_*^3 \quad \text{for} \quad R_* \to 0, \qquad v_* \sim R_*^2 \quad \text{for} \quad R_* \to \infty$$

The case $g < 0$ corresponds to that of a drop clinging to the lower surface of a horizontal wall. The solution of Equation (2.2) which we are seeking and the relationship between the volume and radius of the drop in this case are of the form

$$f(r) = \frac{\gamma\sigma^{1/2}}{(-\rho g)^{1/2} J_1(R_*)} \left[J_0\left(r\sqrt{-\rho g/\sigma}\right) - J_0(R_*)\right]$$

$$R_* = R(-\rho g/\sigma)^{1/2}, \qquad v_* = (-\rho g/\sigma)^{3/2}(v/\pi\gamma) \tag{2.5}$$

$$v_* = \frac{2R_* J_1(R_*) - R_*^2 J_0(R_*)}{J_1(R_*)}, \qquad v_* \sim {}^1/_4 R_*^3 \quad \text{for } R_* \to 0$$

Clearly (see upper curve in Fig.3), $v_* \to \infty$ as $R_* \to \mu$, where $\mu = 3.8317$ is the first positive root of Equation $J_1(\mu) = 0$. Thus, the dimensionless radius of the drop in the case $g < 0$ cannot exceed μ , and with complete wettability ($\gamma = 0$, $v_* = \infty$) we have $R_* = \mu$. This result is cited in [5]. Eliminating γ and v_* , from (2.5) for $\gamma = 0$, we obtain

$$f(r) = \frac{\rho g v}{\pi\sigma\mu^2 J_0(\mu)} \left[J_0\left(r\sqrt{-\rho g/\sigma}\right) - J_0(\mu)\right].$$

In the case of weightlessness ($g = 0$) from (2.2) we have

$$f(r) = \gamma(R^2 - r^2)/2R, \qquad v = {}^1/_4 \pi\gamma R^3$$

The exact shape of the drop in this case is a sphere. (2.1) can easily be used to solve other static problems as well.

We note that Equation (2.1) cannot be used if the angle between the free surface and the horizontal plane is anywhere large. Such is the case, for example, for a drop with $g < 0$ and a sufficiently large v . The shape of the drop in this instance is determined by numerical integration of the exact nonlinear equilibrium equation [5 and 6]. The exact dependence $v(R)$ for $\gamma = 0$ given in [5]. For sufficiently large v no equilibrium forms exist at all. To investigate the stability of the free surface [7] it is also necessary to proceed from the exact solution of the nonlinear problem.

3. Let us consider the small vibrations of a fluid layer of constant depth over a plane floor. We set $h \equiv 0, f = H + \psi$ in system (1.12) and linearize it, assuming $\mathbf{v}$ and ψ to be small,

$$\mathbf{v}_t + \nabla(g\psi - \sigma\rho^{-1}\Delta\psi) = 0, \quad \operatorname{rot}\mathbf{v} = 0, \quad \psi_t + H\operatorname{div}\mathbf{v} = 0 \tag{3.1}$$

Applying the operation div to the first equation of (3.1) and elininating $\mathbf{v}$ with the aid of the third equation of (3.1), we arrive at a single equation for the function $\psi(x, y, t)$

$$\psi_{tt} = gH\Delta\psi - (\sigma H/\rho)\Delta\Delta\psi \tag{3.2}$$

For $\sigma = 0$, $g > 0$, Equation (3.2) becomes a wave equation; for $g = 0$ it becomes the familiar equation of the vibrations of an elastic plate (e.g. see [8]).

Equation (3.2) admits of solutions on the form of plane progressive waves of the form $\psi = A\sin(kx - \omega t)$, where A is the amplitude, ω is the wave frequency, and k is the wave number. The direction of propagation of the wave is assumed to be the x-axis. Substituting this solution into Equation (3.2), we obtain the relation between ω and k

$$\omega^2 = k^2 gH + k^4 \sigma H / \rho \qquad (k = 2\pi / \lambda = \omega / v_0) \tag{3.3}$$

where λ is the length and v_0 is the velocity of propagation of the wave. Formula (3.3) can be obtained by taking the limit as $H \to 0$ of the analogous expression for small capillary-gravitational waves on the surface of a fluid of arbitrary depth [8].

4. Let us investigate the stationary motions of a fluid over a plane floor in the nonlinear formulation. Setting $h = 0$, $u = u(x)$, $v = 0$, $f = f(x)$, we make (1.11) yield the system of equations

$$uu_x + gf_x - \sigma\rho^{-1} f_{xxx} = 0, \qquad (fu)_x = 0$$

Integrating once and eliminating u, we have

$$(q^2 / 2f^2) + gf - \sigma\rho^{-1} f_{xx} = C_1, \qquad fu = q \tag{4.1}$$

Here the fluid expenditure $q \neq 0$ and C_1 are arbitrary constants. Multiplying the first equation of (4.1) by f_x and integrating once more, we obtain

$$(f_x)^2 = [\rho / (\sigma f)]\, P(f)$$

$$P(f) = gf^3 - 2C_1 f^2 - 2C_2 f - q^2 \tag{4.2}$$

where C_2 is an arbitrary constant. Thus, the function $f(x)$ is determined by a quadrature.

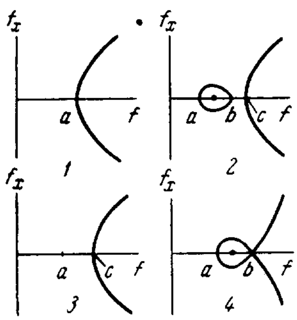

Fig. 4

Let us consider the possible types of stationary motions for $g > 0$, $\sigma > 0$. Since $P(0) < 0$, $P(\infty) = +\infty$, the polynomial $P(f)$ can have either one or three real positive roots. Each such root can have a corresponding progressive flow $f = \text{const}$. The roots a, b, c of the polynomial $P(f)$ are related by the expression $abcg = q^2$. Depending on the values of the arbitrary constants, the following cases are possible:

1) The polynomial $P(f)$ has one positive root $a > 0$. Relation (4.2) between f_x and f is depicted in Fig.4, where the number of each curve corresponds to the number of the case under consideration. The free surface $f(x)$ has the minimum $f = a$ to either side of which it is symmetrical, increases monotonously and tends exponentially to ∞ as $x \to \pm\infty$.

2) The polynomial $P(f)$ has three distinct positive roots $c > b > a > 0$. Two different types of solutions are possible here (Fig.4): an unbounded solution with a minimum $f = c$ similar to Case 1, or a solution periodic with respect to x corresponding to the closed curve in Fig.4. The periodic solution describes steady-state nonlinear periodic waves, where a is the minimal height of the free surface (wave through) and b is its maximal height (wave crest). The wave is symmetrical in shape, both with respect to the trough and the crest, the curvature of the wave being larger in the trough than at the crest. For given ρ, g, σ and expenditure q there is a two-parameter family of such waves (with the parameters C_1, C_2 or a, b).

3) The ploynomial $P(f)$ has multiple roots and $c \geqslant b = a > 0$. The closed curve of Case 2 here converges to a point (Fig.4) describing the progressive flow $f \equiv a$. In addition, as in Cases 1 and 2, there exists an unbounded solution.

4) The roots of the polynomial $P(f)$ are subject to the condition $c = b > a > 0$. The branches of curve 4, Fig.4 to the right of $f = b$ have corresponding monotonous unbounded solutions in which f varies from b to $+\infty$. The loop in curve 4 to the left of the point $f = b$ describes a solitary wave of the trough type. If the trough (the minimum of the function $f(x)$) is at the point $x = x_0$, then $f(x_0) = a$, the function $f(x)$ is

symmetric with respect to the point $x = x_0$ and increases monotonously with increasing $|x - x_0|$; moreover, $f(x) \to b$ as $x \to \pm\infty$. For $x \to \pm\infty$ the flow becomes a progressive flow with a velocity $v_0 = q/b$ equal, by virtue of the relation $ab^2 g = q^2$, to the velocity $(ga)^{1/2}$ of long gravitational waves in a channel of depth a.

Let us now pass on to a coordinate system in which the fluid is at rest at infinity. For a given depth b of the fluid at infinity, solitary waves of the trough type which form a one-parameter family can propagate along its surface. The velocity of the waves is related to the parameter a (minimal depth of fluid) by the expression $v_0 = (ga)^{1/2}$. We note that in contrast to solitary waves in the absence of surface tension [1 and 2], the solitary waves in the case we are presently considering are obtained already in the first approximation of the "shallow water" theory.

The cited solutions differ from the ones published earlier [3] by the fact that they were obtained from the first approximation of the "shallow water" theory, but without the imposition of limitations on the amplitude of the wave and on the relationship between the gravity and surface tension forces.

For the case of solitary wave (x_0 is the trough coordinate), we set

$$P(f) = g(f - a)(b - f)^2, \qquad q^2 = ab^2 g$$

$$x - x_0 = [\sigma/(\rho g)]^{1/2}\xi, \quad f = b\eta \qquad (0 < a/b = m \leqslant 1)$$

in (4.2).

We then obtain

$$(d\eta / d\xi)^2 = \eta^{-1}(1 - \eta)^2(\eta - m), \qquad \eta(0) = m \tag{4.3}$$

for the dimensionless shape of the wave $\eta(\xi)$.

The shape of the wave is determined from (4.3) by integration

$$e^{|\xi|} = \left(\frac{\sqrt{\eta} + \sqrt{\eta - m}}{\sqrt{\eta} - \sqrt{\eta - m}}\right)\left(\frac{\sqrt{\eta(1 - m)} - \sqrt{\eta - m}}{\sqrt{\eta(1 - m)} + \sqrt{\eta - m}}\right)^{1/\sqrt{1-m}}$$

The results of numerical calculation of the wave shape in accordance with (4.3) are depicted on Fig.5, where the numbers by the curves stand for m; the curves are symmetric with respect to the axis $\xi = 0$.

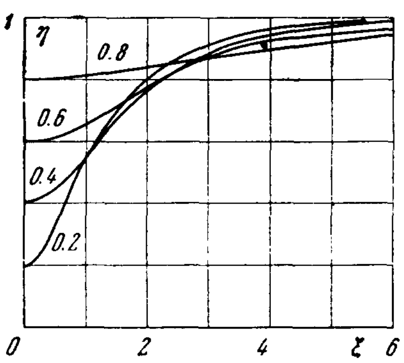

Fig. 5

For $\sigma = 0$ the types of waves described above are not present. In the case of weightlessness ($g = 0$), depending on the constants C_1, C_2, the polynomial $P(f)$ can have from zero to two positive roots. Equation (4.2) then has either no solutions at all, or an unbounded solution (analogously to Case 1), or else a solution of the periodic wave type. Moreover, probressive waves can correspond to positive roots of the polynomial $P(f)$.

In conclusion we note that the solutions of Equations (1.12) approximate the exact solutions of initial system (1.1) only if the assumptions of the "shallow water" theory are fulfilled for them. The method used to construct the solution (Formulas (1.3),(1.5)) implies that the functions f, h and their derivatives with respect to t, x, y must be small quantities of the order of $\delta = \varepsilon^{1/2}$. Here δ is the ratio of the characteristic depth of the fluid to a characteristic dimension in the xy-plane. Thus, a solution describing a solitary trough-type wave is valid provided that $b \ll [\sigma/(\rho g)]^{1/2}$. The wave amplitude characterized by the number m can be arbitrary $(0 < m \leqslant 1)$.

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